# Professor Zygmunt Zahorski's "lecture" on derivatives – prepared for publication by Roman Wituła, the turmoil maker

Zygmunt Zahorski
Roman Wituła – comments and remarks

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## Introduction

Problem, or rather a question which I posed to Professor Zahorski in the winter 1986/87 concerned the set-theoretic and topological "description" of derivatives, quite generally understood by me. At that time I was a newly fledged master in mathematics and my knowledge in this matter was rather "conventional".

After a short period of time Professor gave me in response a letter which I wanted finally to share with others, in the hope that the Readers will be as curious about its contents as me at that time.

**Remark.** In margins of respective pages one can find some comments completing and updating selected parts of Professor's letter. Additionally, after Professor's letter a collection of few longer pieces of supplementary information and the bibliography are given.

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## Professor's letter

See paper [56].

Well, my paper published in Transaction of the American Mathematical Society in 1950, but prepared actually in 1939-42, starts with the following words: "characterization of the class of continuous functions, possessing everywhere the first derivative, with the aid of their topological and metrical properties is unknown". It is not important whether the continuous function or its derivative (in general, discontinuous in many points) will be characterized. I aimed at characterization of derivative existing everywhere. On the contrary as in case of analytical functions, in this case the unbounded functions are more difficult, therefore I put the following stages: I – bounded derivative, II – finite derivative, III – infinite (in some points) derivative – for a long time (however $> 1900$) there is known a proof that the set III is a null set and, which is more, it happens without the assumption of derivative existence (in other points) and even the continuity is not needed and it is enough to consider the one-sided derivative: the set of all points in which the right-sided infinite derivative of any function (even the unmeasurable function) exists is the null measure set.

This is the Banach result [7].

Proof of this fact results, for instance, from the original Weierstrass' proof of this function nowhere-differentiability, presented for the first time in the Weierstrass' letter to Du Bois Raymond (1875). Essential in this matter seems to be also the Banach theorem (1931) which will be cited in third item of supplementary information given at the end of this paper.

In case of the continuous nowhere differentiable Weierstrass function in each point one of the Dini derivatives is $= +\infty$, the other is $= -\infty$, but the Dini derivatives are not the derivatives, similarly as the upper and lower limits of a sequence are not (in general) the limit of this sequence.

Considering the derivative existing almost everywhere (i.e. except the $(\mathcal{L})$ null measure set of points), the problem is easier and was solved in 1912 (and published in 1915 or 1916) in the Luzin's PhD dissertation in Russian ("Integral and trigonometric series") – to be more precise one should note that PhD degree was in Russia, even in the tsarist times, the second kind scientific degree corresponding with our habilitation. In any case, the postdoctoral dissertations are different everywhere, even in the same university – this one was of epochal matter considering not only this one problem, but also several other problems put there. At least one Luzin's hypothesis (claiming that the Fourier series of square integrable function is convergent almost everywhere) appeared to be unlikely true, after Kolmogorov's examples of the Fourier series of a function integrable in the first power, divergent almost everywhere, presented in 1922 and published in Polish journal Fundamenta Mathematicae probably in 1923 and such defined series for the $L^1$ function, divergent everywhere, presented in 1926 and published in C. R. Acad. Sci. Paris in the same year. The outstanding authority in the matter of trigonometric series, Professor Antoni Zygmund, said in 1960 that it is certainly false since in $L^2$ the convergence in $L^2$ is typical (Riesz-Fischer Theorem, 1904), not the almost everywhere convergence. Before 1936 Menshov gave the examples of orthogonal bounded systems with the almost everywhere divergence in $L^2$ and Kolmogorov put in 1926 something more than hypothesis, published in 1927 in German journal Mathematische Zeitschrift (in paper joint with Menshov), saying that it can be even the trigonometric system, but properly rearranged. Namely he wrote that he could give such example, however he has never given, there or anywhere else, any proof nor function – or, which is equivalent, any coefficients of such series nor the way of rearranging the terms (permutation in the infinite sequence). Many mathematicians (Russians, Hungarians, Americans and probably the others) tried to do that. I succeeded in 1960, maybe after three weeks of work (and a little bit in 1954), and the many-years unsuccessful efforts to prove the Luzin's hypothesis appeared to be

See papers [33, 55]

very useful for this achievement. In 1954, quite quickly as well, I thought that I had a construction, however I noticed a mistake right before giving a talk for the Polish Mathematical Society. I gave indeed the talk, but on the completely different and known, however not to all of the audience, subject – on the so called Young Theorem about symmetry (for any function of point on the $x$ axis of numerical values the sets of right-sided limits of $f(x_n)$, for $x_n \to x$, $x_n > x$, and left-sided limits for $x_n < x$, are identical and one of them is equal to $f(x)$. Everywhere? Not necessarily, however with the exception of the at most countable set of $x$s – which is better than the null set.)

Probably it concerns Mrs G.C. Young, see [47, 17].

Abbreviated, but still clear for the specialists, proof, with a function and permutation certainly, I submitted in 1960 to C. R. Acad. Sci. Paris (they publish within three weeks but only 1–5 pages, notices with no proofs or with very shortened proofs).

See paper [58].

I said to Professor Zygmund, who was then for few days in Warsaw (he lives in Chicago) that since 1945 I believe in the Luzin's hypothesis and I have no doubts caused by the truth of Kolmogorov's hypothesis that it can be different for the rearranged systems, whereas for the normal order $0, 1, 2, 3, 4, \ldots$ it is exactly as predicted by Luzin. I announced this in 1961, in C. R. Acad. Sci. Paris as well, but before final elaborating and reporting it in the Institute of Polish Mathematical Society in Warsaw for Professor S. Mazur – he passed away in November 1981. During the edition I have found an error. And, since the note was already published in C. R., I announced in Mathematical Reviews, through the mediation of Professor Zygmund, that there is no solution, a mistake. Nevertheless, the Luzin's hypothesis appeared to be true and has been proven in 1966 by Swedish mathematician L. Carleson, I think about 10 years younger than me. Supposedly he worked about 7 years in good conditions – on American scholarship at the Stanford University in California. He gave one of the main talks at the International Congress in Moscow in 1966, chairman for this talk was Kolmogorov. Luzin did not live to see the proof of his hypothesis, he passed away in Moscow on 28 February 1950.

See paper [18].

And this is the Luzin's result (construction proof was given in his above mentioned work – reprint, containing various comments and works of authors giving the solutions of some Luzin's problems or similar problems as well, was made in 1950. I had this book but I lost it somehow during my move to Gliwice in 1970).

See paper [39].

*It is necessary and sufficient for function $g(x)$ to be almost everywhere the derivative of a continuous function, that $g(x)$ is $(\mathcal{L})$ measurable and almost everywhere finite.*

Proof of this theorem can be also found in monograph [14].

However Luzin does not call this continuous function as antiderivative nor indefinite integral of $g$. It is because we have here a high rank of uncertainty – different "antiderivatives" do not differ in constant. Indeed, he constructs one of the "antiderivatives", but it is not at all unique. He uses here his theorem, in which he claims that for a function measurable on the interval there exists a closed set of the measure differing less than $\varepsilon$ from the length of this interval (certainly, in general smaller than this length) where $f$ is relatively continuous. One can say, by omitting the isolated points, that this is the perfect set, i.e. closed and dense-in-itself.

First example of functions, difference of which is not constant on the given interval even though they have everywhere in this interval equal derivatives, was given by Hans Hahn [27]. Another example presented S. Ruziewicz (see [45, 46]).

Somebody has presented this result on my seminar at the University of Łódź, however I do not recall too much of it and even if I could reconstruct this Luzin's "antiderivative", it would take probably about three months of my good work. It is done by the use of some functions defined on sets similar to the Cantor set, but of positive measure. Necessity of these conditions is relatively simple: it is finite almost everywhere because the derivative, as I have noticed above, can be $+\infty$ or $-\infty$ only in the null measure set. Measurability almost everywhere of the derivative of a continuous function (and even each of four Dini derivatives) is quite easy to prove, probably even without the assumption about continuity. I wrote something about that in Annales of Polish Mathematical Society in 1952, unfortunately I do not remember too much, however it was probably new at that time. By the way: even if the derivative exists everywhere, then if for example it is equal to $+\infty$ on the set of cardinality of the continuum, it is hard to speak about the antiderivative, for example, it is quite easy to construct two continuous functions possessing everywhere equal derivatives, finite outside of the Cantor set and equal $+\infty$ on the Cantor set, not differing in a constant. Every generalization of antiderivative function, for example presented in the second part of "Outline of the Theory of Integral" by Saks [47] (Polish edition from 1930, French – translation differing in only one chapter, right in the middle, and English edition, probably from 1937, completely different and more extensive, not known closely to me since I do not speak English), assumes always that the derivative is finite almost everywhere, it means with the exception of at most countable set, and the expression almost everywhere means with the exception of the null measure set, which obviously can be uncountable or even of cardinality of the continuum. Not because the derivative would must be finite almost everywhere, like it can be seen in the mentioned example with the Cantor set, only because in the other case it is hard to talk about the antiderivative functions differing in a constant. This is the sufficient condition and I do not know any other less inconvenient.

Probably paper [57] is concerned.

English version of Saks' monograph is [47]. Professor Zahorski refers here certainly to the Polish version [48] of this book.

If the derivative is finite, then for finding the antiderivative the Lebesgue integral is sufficient. Even if the derivative does not exist on the set of cardinality of the continuum, it is only needed that it exists almost everywhere, since the function satisfying the Lipschitz condition – or even more, the difference of two monotonic functions, continuous or not – possesses almost everywhere finite $(\mathcal{L})$ integrable derivative, even though this integral (as the function of upper limit) in general differs from the differentiable function by two elements – the so called discontinuity function – they are of the first kind here and only for such ones it is defined, and – even if it is continuous – the so called singularity function. Only if it is absolutely continuous (it does not concern $|f|$, even though if $f$ is absolutely continuous, then $|f|$ as well), then the singularity function is equal to 0 for each $x$. But the Lipschitz functions are absolutely continuous, the integral of a measurable bounded function certainly satisfies the Lipchitz condition and each measurable bounded function is also $(\mathcal{L})$ integrable (over intervals of finite length), thus everything is correct.

Some unbounded functions are also $(\mathcal{L})$ integrable, of course functions from among $(\mathcal{L})$ measurable functions, and always, no matter if $f$ is bounded or not, and even for functions infinite on the set of cardinality of the continuum and null measure (if the set would be of measure $> 0$, then the $\mathcal{L}$ integral would not exist), here is $\frac{d}{dx}\int\limits_a^x f(t)dt = f(x)$ almost everywhere. First part of the above mentioned "Outline of the Theory of Integral" by Saks concerns the $(\mathcal{L})$ integral. Second part is devoted to the Perron and Denjoy integrals, the more general integrals than the Lebesgue integral. Because, unfortunately, when $f'(x)$ is unbounded, even if it exists and is

Professor Zahorski refers certainly to the Polish version of this book (see [48]).

finite everywhere (measurable, of course), it can be not ($\mathcal{L}$) integrable. Then, considering the definite integrals $\int_a^x f'(t)dt = F(x)$, the problem is solved only by the higher Denjoy integral, equivalent to the Perron integral, proof of which can be found in the second part of Saks' "Outline". Both these integrals have been defined not just independently, but completely differently and only later the other authors proved two theorems – the first, I think, that Denjoy's definition (more specific) contains the Perron's definition, and vice versa which is claimed by the second theorem. Denjoy's descriptive definitions are similar to the Lebesgue's definition, with such difference that instead of the idea of absolutely continuous (AC) function the idea of absolutely continuous generalized (ACG) function is applied in the definition of the narrow $\mathcal{D}$ integral and the idea of absolutely continuous generalized in the wider sense function (ACG$^*$) is used in the definition of the wide $\mathcal{D}^*$ integral. In $\mathcal{D}$ definition the approximative derivative is used, whereas $\mathcal{D}^*$ definition uses the ordinary derivative (for $*$ it "should" be more weird, here it is traditionally in opposite). However, the descriptive definition is not much worth without the constructive definition, because it does not guarantee the existence of defined elements. In case of $\mathcal{D}$ and $\mathcal{D}^*$ integrals the constructive definition is "terrifying". There are actually the constructions of integrals of more and more high ordinal classes, finite or countable (similar to the hierarchy of Baire's functions and Borel's sets) starting with the Lebesgue integral as number 0. So, transferring to higher classes is executed through: 1) the Cauchy process of creating the improper integrals for isolated points. It is a well known fact that the $\mathcal{L}$ integral includes the $\mathcal{R}$ integral. But only the proper integral or the improper absolutely convergent integral. Conditionally convergent improper $\mathcal{R}$ integral comes beyond the $\mathcal{L}$ integral (but is included in the first class of $\mathcal{D}$ integrals); 2) the Harnack process – which I prefer to omit, when the integrability in the lower class is disturbed by some perfect nowhere dense set, such that the integrability occurs in the intervals of its compliment, whereas in the entire line segment (composed of these intervals plus this nowhere dense set) the integrability does not occur. But in this set itself the integrability occurs as well. Perron treated this in the completely different way, by means of the so called raising and descending functions, probably something analogous exists for the ordinary differential equations, in which he was involved as well. Besides, seeking the antiderivative is the simplest version of a differential equation – only with a number of other complications, therefore to avoid the situation of too big amount of complications the continuity is assumed, not nicely said, so the solution from $C_1$ class with the continuous derivative is sought.

The Denjoy integral is called by Russians the Chinczin integral. Chinczin gave such definition almost in the same time, in about 1916. In foreign spelling the name is Khintchine, devil knows of which nationality. In Russian it is simply Хинчин, but I do not know whether the meaning of this word is the same as in Polish, since the Chinese in Russian sounds as "kitajec".

The name Chinczin sounds as "Chińczyk", which in Polish language means the Chinese.

Perron and Denjoy (narrow) integrals serve not only for seeking the antiderivative functions for derivative existing everywhere or (in ACG class) almost everywhere, besides the measurable non-integrable functions exist even in the wider Denjoy sense, but for seeking definite integrals of functions integrable in this sense, and $\frac{d}{dx}\int_a^x f(t)dt = f(x)$ almost everywhere for the narrow integral, approximative $\frac{d}{dx}$ for the wide integral, both give for $\int_a^x f(t)dt = F(x)$ the ordinary continuity with respect to $x$, and even more – ACG or ACG$^*$. The wide Denjoy integral gives the antiderivative function for the approximative derivative, which is better to omit.

Incidentally, Denjoy has defined some definite integral coming beyond $\mathcal{D}$ and $\mathcal{D}^*$, particularly for the trigonometric series. I do not know this definition, I have never seen it and I doubt whether I could solve this problem (having probably more than one solution – for example, whether each wide definition would be good enough, the narrow definition rather not, however even this is questionable). This problem has been mentioned also by Luzin in his PhD dissertation, but Luzin did not solve it and maybe - I do not know exactly - he only schematically specified it. It is about the following:

Let us assume that the trigonometric series $\frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$ is everywhere convergent to a finite function $f(x)$, certainly of the first Baire class. According to the theorem proved by Cantor in 1870 roku, coefficients $a_n$, $b_n$ are uniquely determined by function $f$ (more precisely, Cantor proved that if $f(x) = 0$ for each $x$, then all $a_n, b_n = 0$, which is certainly equivalent – proof of this theorem, although is called as "elementary", is not so easy and can be found, for example, in "Principles of differential and integral calculus" by S. Kowalewski, German, translated into Polish by I. Roliński. When in 1928 in 7th grade of primary school I was reading Kowalewski's book, I could not know that I will meet, the deceased already, Roliński personally after 1948 in Łódź. He was the honored teacher in secondary schools, the popularizer and, as one can see, the translator, after war the professor in Pedagogical Academy in Łódź and after joining it with University of Łódź he became the professor of this university. He did not have his own results but he was a really cool fellow, level-headed and wise, with the sense of humor and good manners, not despotic to students and workers of lower rank. He was also the brother-in-law of Rozwadowski who was the bishop of Łódź at that time). Problem: how to define an integral, such that coefficients $a_n$, $b_n$ could be determined by using the known Euler-Fourier formulae with the integral according to this definition?

Author of these "Principles ..." is Gerhard Kowalewski (1876-1950). See also item 4 of supplementary information.

After Cantor, the uniqueness theorem was generalized, first in a case when $f(x) = 0$ with the exception of at most countable set, it was obvious that one could not resign from the set of measure $> 0$, but for the null measure sets some troubles arose – for some, the so called $U$, the theorem was true, for others, the so called $M$, the theorem was false.

See also item 5 of supplementary information.

Many works concerned the sets $U$ (unicite – uniqueness) and $M$ have been written by A. Rajchman, docent in the University of Warsaw, killed in Dachau probably in 1941, Zygmund, Mrs N. Bari, Russian with French or Italian surname, the author of monograph devoted to the trigonometric series, competitive with the Zygmund's monograph, but in Russian – I do not know whether Americans translated it (Bari in 1961 in the age of 60, being almost blind but wanting often to walk without assistance, during the (domestic) Forum of Mathematicians in Leningrad, got run over by a tram or electric train.) and, first of all, Menshov, I think he still lives but is at least 85 years old.

For sets $U$ and $M$, we propose to take a look into [59, 9] and [32].

There exists an English translation [9].

It is known that if $f$ is $(\mathcal{L})$ integrable, then the coefficients are expressed by means of $(\mathcal{L})$ integrals, thus, in particular, if $f$ is $(\mathcal{R})$ integrable, then by means of $(\mathcal{R})$ integrals.

But what if $f$ is not $(\mathcal{R})$ integrable? Denjoy gave in about 1923 the required definition of integral (and he proved the formulae). Whether they concern also the exception of countably many points – I do not know, and the uncountable would threaten with something maybe worse than the $M$ sets, which does not mean that one cannot try. I suppose that till today the characterization of $U$ (null measure) sets is not known or, what is equivalent, of the $M$ (null measure) sets. I did not deal with this subject by myself.

Euler himself proved his formulae, but in completely bad way, what one cannot have any grudge against him for, because there was no proper definition of an integral, or even of a derivative (equal to the quotient of "infinitely small" increments, it was said at that time, not using the concept of limit), or sufficiently wide concept of a function (they were introduced only in 1837 by Dirichlet and supposedly at the same time by Lobaczevski) – the first proper definition of a definite integral was given by Cauchy, and he has done this only for continuous functions, after 1800, and Riemann transformed it, in about 1850, for these discontinuous functions, for which it could be applied to, characterized since Lebesgue's days as bounded and continuous almost everywhere. Euler was precise in works on the integer numbers theory, however in analysis, not precise by necessity these days, he did a lot and he had a good nose for it – he obtained results correct in general, despite some inaccuracies, <u>first-class intuition</u>. In view of incorrectness of his proof, the trigonometric series theory went in two directions: 1) resignation of the proof, which can be correctly done by transferring to definition: series with coefficients computed in such a way with the aid of function $f$ <u>called</u> the Fourier series of function $f$ (the names are traditionally undeserved – the Fourier series have been introduced by this author in 1822 in the book concerning the heat conduction equation (partial differential equations), d'Alembert before 1800 considering (partial differential equation as well) the vibrating string equation, and Euler in about 1750 considering the periodic phenomena, for example astronomical, it means that Euler was the earliest, Fourier the latest), not taking care (as Euler wanted) if it is convergent and if exactly to $f$. It does not mean that the problem was get off lightly, it was only transformed to some other problem – one can investigate later if it is convergent to $f$, and even if it is not, how $f$ can be found by using it – for example, it turned out (Lebesgue's proof) that the so called first arithmetic means of partial sums converge almost everywhere to $f$ and, which is more, in $L^1$ – Fejer proved this earlier in the continuity points and the uniform convergence of the first means on the whole $x$ axis in case when $f$ is continuous everywhere and periodic with period $2\pi$. Both of these theorems make a part of elementary analysis, the thing is that some functions from $L^1$ can have not a one continuity point, but almost all points are their so called Lebesgue points – I omit the definition, and in these points exactly this convergence holds. In $L^2$ (for each orthogonal expansion, not only the trigonometric one) the convergence $\to 0$ holds in the sense of integral distance, i.e. in the metric of Hilbert space $L^2$. Then admittedly the sequence of partial sums is only convergent in measure (I omit the definition) and not almost everywhere, but the subsequence convergent almost everywhere may be selected – in the trigonometric system it has been known for a long time that $S_{2^n}$ is enough, now (since 1966) we know that $\ldots S_n$ itself, meaning the whole sequence $S$.

See item 2 of supplementary information.

Very good positions referring to the history of Fourier series and the "priority" problem are papers [28, 12, 20] and [60].

It concerns the L. Carleson result [18]. More information can be found in [29] (see also [10, 11, 26, 22]).

Second direction went into the uniqueness. Yet Riemann did practically correct proof of the Euler theorem and for the simplest function, everywhere equal to "0", because even for this function the proof was missing and the one made by Euler would be incorrect here as well. Last word till now is given by the mentioned Denjoy's work. He died recently at the age of 90. I saw him lately in Bulgaria (Varna) in 1967.

$\mathbf{M_k}$ and $\mathcal{M}_\mathbf{k}$ classes are called, which is obvious, the Zahorski classes; and Professor Zahorski used letter $\mathbf{M}$ (respectively $\mathcal{M}$) from the initial of name of his sweetheart at that time.

Beautiful application of the Darboux property for derivatives can be found in paper [54], where the "equivalence" between the fundamental theorem of integral calculus and the Lagrange mean value theorem is proved.

Let me return to the actual subject. Well, in paper [56], mentioned at the beginning, I defined six classes of sets $\mathbf{M_k}$, $k = 0, 1, \ldots, 5$, five classes of functions $\mathcal{M}_\mathbf{k}$, $k = 1, 2, \ldots, 5$ and class $\mathbf{J}$ = functions of the first Baire class taking all the intermediate values in each interval, which is called the Darboux property. This property is possessed (elementary analysis) by all the continuous functions but not only – it is also possessed by the approximatively continuous functions and also not only – by the derivatives existing everywhere, even if they are not approximatively continuous, as well. Proof for the derivatives is easy. It is sufficient to prove that if $f'(a) > 0$, $f'(b) < 0$, (or conversely), $a < b$, then there exists $\xi \in (a, b)$ such that $f'(\xi) = 0$. This is the complete analogy of the Rolle Theorem, so it is strange that many manuals for elementary analysis omit this fact. In considered case the absolute maximum of $f$ in $[a, b]$ should be taken, but it cannot be taken in $a$ or in $b$ either, therefore in $\xi$ within the interval. But then $f'(\xi) = 0$ (often called as the Fermat Theorem in analysis – certainly his theorem saying that each natural number is a sum of at most four squares of natural numbers is much more spectacular). In the opposite case, the absolute minimum should be considered.

Equally easy is belonging to the first Baire class. If $f'(x)$ exists everywhere then $f'(x) = \lim\limits_{n\to\infty} n(f(x+\frac{1}{n})-f(x))$ and functions $f_n(x) = n(f(x+\frac{1}{n})-f(x))$ are, for each fixed $n$, continuous. These two necessary properties of the everywhere existing derivative have been certainly known for the long time. Lebesgue gave a very simple example that they do not characterize the derivative: functions $f(x) = \begin{cases} 1 & \text{for } x = 0 \\ \sin\frac{1}{x} & \text{for } x \neq 0 \end{cases}$, $g(x) = \begin{cases} 0 & \text{for } x = 0 \\ \sin\frac{1}{x} & \text{for } x \neq 0 \end{cases}$ are both of the first Baire class with the Darboux property, so if they would be the derivatives (bounded, as it can be seen), then the function $f(x) - g(x) = \begin{cases} 1 \text{ for } x = 0 \\ 0 \text{ for } x \neq 0 \end{cases}$ would be derivative as well, which is impossible, since it does not have the Darboux property. Thus, $f$, or $g$, (or both) is not the derivative. Function $f - g$ belongs to the first class, which obviously must happen. I cited this example because I still use its tiny modifications for creating other (simple, as well) counterexamples. Direct proof of the fact that $f$ is not a derivative would be not much difficult, but what for. Well, for a long time not very difficult theorem saying that $f$ belongs to the first class if for each $a \in \mathbb{R}$ sets $\{x : f(x) > a\}$ and $\{x : f(x) < a\}$ are of the $F_\sigma$ class has been known. It is even enough to know that for the set of value $a$, dense on the $y$ axis, even countable. Therefore the sets of all my classes belong to the $F_\sigma$ class and the empty set I include to all of them, in order to avoid the exceptions caused by it (one cannot contradict that it does not belong, the so called truth in empty way). The definition is not trivial only if the set is nonempty. We have $\mathbf{M_0} \supset \mathbf{M_1} \supset \mathbf{M_2} \supset \mathbf{M_3} \supset \mathbf{M_4} \supset \mathbf{M_5}$ and all these inclusions denote proper subsets (which I say before the definition, but which can be justified only after the definition). Definitions: $E \in \mathbf{M_0}$ if each point $x \in E$ is the both-sided limit point for $E$; $E \in \mathbf{M_1}$ if each point $x \in E$ is the both-sided condensation point for $E$, that is,

See for example [38].

in each one-sided neighbourhood $x$, $(x-\delta, x)$ and $(x, x+\delta)$, $\delta > 0$ the uncountable part of set $E$ is included (it means of cardinality of the continuum, since it is the Borel set); $E \in \mathbf{M_2}$ if this part is of the measure $> 0$; $E \in \mathbf{M_3}$ (it can be easier, however it is done like that for the purpose to enclose also $\mathbf{M_4}$ where complications cannot be avoided – I am full of doubts whether I could recall these few quantifiers, even though I invented this condition on my own, and I rather do not feel like approaching the bookcase for reprint, however I will try without the reprint – if there exist the sequences: of closed sets $\{F_n\}$ and of numbers $\{\eta_n\}$, $\eta_n \geqslant 0$, such that for every $x \in F_n$ and $\varepsilon > 0$ there exists $\delta > 0$, such that if $hh_1 > 0$, $|h + h_1| < \delta$, $\frac{h}{h_1} < \varepsilon$ then $\frac{|(x+h, x+h+h_1) \cap E|}{|h_1|} > \eta_n$ ($|\cdot|$ in numerator denotes the Lebesgue measure), interval $(x+h, x+h+h_1)$ is written here without the usual agreement that the first number denotes the left bound; it is like this if $h > 0$, in opposite if $h < 0$, but for not making the exceptions in the entire paper this agreement is not applied). I will not bet if it is like this, or equivalently, or completely wrong – I just want to show complexity of this condition; $E \in \mathbf{M_4}$ if $E \in \mathbf{M_3}$ and for each $n$, $\eta_n > 0$; $E \in \mathbf{M_5}$ if each point $x \in E$ is its density point, which means that $\lim\limits_{h \to 0^+} \frac{|(x-h, x+h) \cap E|}{2h} = 1$.

I include function $f$ to class $\mathcal{M}_\mathbf{k}$, if for each $a \in \mathbb{R}$ sets $\{x : f(x) > a\}$ and $\{x : f(x) < a\}$ belong to $\mathbf{M_k}$. Unlike for the sets, I prove that $\mathcal{M}_\mathbf{0} = \mathcal{M}_\mathbf{1} = \mathbf{J}$ (let us recall that class $\mathbf{J}$ is the first Baire class with the Darboux condition), therefore class $\mathcal{M}_\mathbf{0}$ is redundant. Certainly $\mathcal{M}_\mathbf{k} \supset \mathcal{M}_\mathbf{k+1}$ and for $k \geqslant 1$ it is the proper subset, on the contrary as for $k = 0$. Well, I proved there that class $\mathbf{M_4}$ precisely characterizes the sets $\{f'(x) > a\}$ (alternatively $< a$) for the bounded derivative, for $\{f'(x) > a\}$ it takes place even just for the bounded from above $f'$ – it is one of two most difficult theorems in this paper. Necessity is even tolerable, but sufficiency – few fine pages with no "waffle", since one can waffle during the lecture or in the unsuccessful, because "over-waffled", script (for external students – I always thought that preparation of external students is miserable, so it is exactly why one should not "waffle" too much to them – these wretches learn by heart and get lost immediately); really good teachers waffle a little bit even in manuals for normal students, for example Mostowski, but as little as possible – however the printed papers I wrote without waffling. It happens that even the 1-page proof is tough (and the 10-page proof can be easier). The main idea is the same as in (not known closely to me) Carleson's paper devoted to the Luzin hypothesis, even though I wrote it many years earlier, I do not know whether he read it, anyway I did not patent this idea, since it is simple and maybe used by somebody else before me. This is the analogy to division of, for example, agricultural property in the successive inheritance cases, for simplification, always by half but satisfying two conditions: 1) one can never divide regions $\leqslant 4$ hectares, 2) one can never divide if profitability, for instance the soil quality, of at least one half would decrease below a constant number predetermined for every divisions. Then, after the finite number of divisions, the region will break down into the parts unequal, in general, with respect to the area and such that none can be divided any more. In my work this rule of division is represented by preserving the mean density of a given set above some number, in Carleson's work by something more complex, however, both of us divide the intervals in half with the aid of a middle point. One can of course consider more than two restrictions of division, but it was not necessary in these papers. Later two estimations resulted in my work – one for these segments for which the second prohibition did not work (maximal numbers of divisions, the shortest possible segments) and the second one for the segments stopped to be divided earlier – each of these sets can be empty, but not both of them. What resulted in Carleson's work – I do not know and I do not want to read it for not making my work harder, since it is already hard enough. None of the "laws" forces me to do this work, but I want to do it. Anyway, from June 1981 till October 1984 I was dealing with something else and something more difficult (I think, because it is older and unsolved till now, however it is not the evidence of its higher difficulty). From October 1984 till February 1986 the trigonometric series again and from February 1986 the other thing – alternately.

What next in this paper? Theorems that the finite $f'$ belongs to $\mathcal{M}_{\mathbf{3}}$, infinite in some points $\in \mathcal{M}_{\mathbf{2}}$, but even in this case the sets $\{a < f'(x) < b\}$ for $a$ and $b$, finite as well as infinite, are in $\mathbf{M_3}$. Thus, all the conditions are stronger than the previously given $\mathcal{M}_{\mathbf{1}}$, but still these are not the characterizations, only the necessary conditions.

See item 6 of supplementary information.

And what about $\mathcal{M}_{\mathbf{5}}$? This is the characterization of $\mathcal{A}$ class – class of the approximatively continuous functions which are, as it can be seen, of the first Baire class, not conversely. This can be added to the package, since it was known – given probably by Maximoff in 1936 in Japanese journal Tôhoku Math. Journal. Does the bounded derivative belong to $\mathcal{M}_{\mathbf{5}}$? As it can be seen, it does not have to. Or conversely, does the function from $\mathcal{M}_{\mathbf{5}} = \mathcal{A}$ must be a derivative? If it is bounded, yes, if unbounded, not necessarily. Moreover, the example patterned on the mentioned Lebesgue's example with $\sin\frac{1}{x}$ (concerned the class $\mathbf{J}$, the widest one) shows that belonging of the <u>bounded</u> function to $\mathcal{M}_{\mathbf{4}}$ class does not guarantee that it is a derivative. That is, $\mathcal{M}_{\mathbf{4}}$ does not give the characterization of bounded derivatives, even though $\mathbf{M_4}$ gives the characterization of sets $\{f'(x) > a\}$ for these derivatives. This implies that the class of bounded derivatives cannot be characterized by layers, with the aid of distribution function, i.e. functions $a \mapsto \{g(x) > a\}$. I know from Professor Lipiński that some American wrote that in this paper the problem of such characterization of bounded derivatives had been posed. Nothing of that kind. Question about "$\mathcal{M}_{\mathbf{4\frac{1}{2}}}$" class concerns some other (I do not know which one) characterization, not by means of distribution function which is written there clearly. It is because (confining to the bounded functions) their belonging to $\mathcal{M}_{\mathbf{5}}$ is sufficient for them to be derivatives, however it is not necessary – this condition is too strong. And belonging to $\mathcal{M}_{\mathbf{4}}$ is necessary, however not sufficient – the condition is too weak (bounded $\mathcal{M}_{\mathbf{4}} \supset$ class of bounded derivatives $\supset \mathcal{M}_{\mathbf{5}}$ with restrictions, both inclusions are proper). Neugebauer, the American, characterized indeed the derivatives (I just do not know whether they were only bounded), but he gave the condition not differing too much from the definition of derivative, that is from the trivial condition mentioned here at the beginning. I saw this, but I do not remember. Professor Lipiński (University of Gdańsk) knows something more about that. Class of bounded derivatives should be $= \mathcal{M}_{\mathbf{4\frac{1}{2}}}$ bounded (since the function belonging to $\mathcal{M}_{\mathbf{5}}$ does not have to be bounded, belonging to $\mathcal{M}_{\mathbf{4}}$ does not have to be either). The $\mathbf{M_{4\frac{1}{2}}}$ sets are not needed of course.

Mentioned here the Neugebauer theorem characterizing the derivatives (with the proof) and many more information in this subject can be found, for example, in monographs [14] and [25]. See also item 6 of supplementary information.

This is the most exhaustive answer I can give you for your question. Anyway for over 30 years I have not dealt with this subject-matter or, in general, with the real functions either. I work on trigonometric series, as a matter of fact on one problem concerning them – the convergence almost everywhere in a set, convergence in a point, even though the known conditions are too strong (sufficient, not necessary ones), it is not worth to deal with them, the convergence "takes place when it takes place", after all too complicated conditions are not worth too much. And sometimes I work on something else, for a rest or for a change, or even because the cat looking after one hole, dies.

Sincerely yours, Z.Z.

## Some additional pieces of information (R.W.)

1. Professor Zahorski's letter perfectly matches with [61] (Scientific Note created on the occasion of 70 anniversary of Professor's birthday, containing, among others, Professor's biography – which is included, in Polish and English version, in this monograph – and list of his works; papers presented there have been prepared by the very noble, international group of Professor's former students, colleagues and continuators of Professor's idea; definitely valuable scientific publication, deserving fame). It also gives the author's take on the concept of Zahorski's classes, on the Carleson theorem, etc. Moreover, this letter completes the opinions on the given issues of the other Polish authority in the field of real functions – Professor Jan Lipiński, included in [61], in his survey.
2. In 2007 the 300 anniversary of L. Euler's birthday was celebrated. This event gave a new occasion to study his works. In connection with this and not only "in this connection", we understand today much better the methods of proving of this genial mathematician. Even though this author seems to be far from the present-day formalism (Professor Zahorski writes about this), many elements of his creation can be translated on the language perfectly correct today, the purely formal as well as by the usage of limits (a good example of this phenomenon can be papers [1, 2, 6, 36, 37]).
3. The Banach theorem, mentioned on the margin of page 82, sounds as follows:

   *Set of all functions $f \in C[0,1]$, for which in each point $x \in [0,1)$ we have*

   $$D_+ f(x) = -\infty \quad \textit{or} \quad D^+ f(x) = \infty$$

   *and concurrently in each point $x \in (0,1]$ we have*

   $$D_- f(x) = -\infty \quad \textit{or} \quad D^- f(x) = \infty,$$

   *is residual.*

   Italian mathematician Pier Mario Gandini, in paper [24], generalized this theorem for spaces $C([0,1]^n)$, additionally extending the adequate residual set to the complement of $\sigma$-porous set.
4. There exists at least one more reason for which one should mention Gerhard Kowalewski. He is the author of the following theorem on the mean value for the system of equations formed from $n$ integrals.

**Theorem ([34]).** *Let $x_1, \ldots, x_n \in \mathbb{C}_{\mathbb{R}}[a,b]$. There exist numbers $t_1, \ldots, t_n \in [a,b]$ and nonnegative real numbers $\lambda_1, \ldots, \lambda_n$ such that*

$$\sum_{k=1}^{n} \lambda_k = b - a$$

*and*

$$\int_a^b x_r(t)dt = \sum_{k=1}^{n} \lambda_k x_r(t_k), \quad r = 1, \ldots, n.$$

In paper[35] Kowalewski generalized this theorem by substituting the linear measure $dt$ by the weight measure $F(t)dt$, where $F \in \mathbb{C}[a,b]$, $F$ is of the constant sign on $(a,b)$ and

$$\sum_{k=1}^{n} \lambda_k = \int_a^b F(t)dt.$$

Only in 2008, Slobodanka Janković and Milan Merkle in paper [31] extended this theorem for any intervals $I \subset \mathbb{R}$ by introducing in place of measure $F(t)dt$ any finite positive measure $\mu$ defined on the Borel $\sigma$-field of interval $I$ (functions $x_k$ belong then to the set $\mathbb{C}_{\mathbb{R}}(I) \cap L_\mu(I)$, $k = 1, \ldots, n$, respectively).
Moreover, as it is noticed by these authors, except two citations the Kowalewski's results remained completely unknown – how unfairly. The above results, together with the proofs, are also presented in monograph [30].

5. History of the uniqueness theorems for trigonometric series (including the multiple trigonometric series) still lasts (see papers [3, 4, 5, 55, 59]). It is worth to mention additionally the Du Bois Reymond result [13] from 1876, a little bit younger in relation to the Cantor result, cited and proved in [59] as well as in [9]:

**Theorem A.** *If trigonometric series*

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} \big(a_n \cos(nx) + b_n \sin(nx)\big)$$

*is convergent everywhere to a finite sum $f(x)$ integrable on $[0, 2\pi]$, then it is the Fourier series of function $f(x)$.*

Let us notice (after Nina Bari [9]) that, originally, Du Bois Reymond referred this result to integrability in the Riemann sense (his result holds true also in the case when we neglect the convergence of the trigonometric series on some countable set). Appropriate extension of Theorem A for the functions integrable in Lebesgue sense we owe to Lebesgue himself (however Theorem A is still called the Du Bois Reymond theorem). Obviously, Theorem A implies the discussed here Cantor's result from 1870.
At the end let us also recall that the similar uniqueness theorems for the Haar and the Walsh series have been discussed and solved, among others, by Russians W.A. Skvorcov (see [50, 51, 52]) and M.G. Plotnikov (see [42, 43, 44]). The subject

of uniqueness of the multiple trigonometric series is also discussed in the third chapter of V.L. Shapiro's monograph [49].

6. The characterization of derivatives in the class of bounded functions $f\colon (0,1) \to \mathbb{R}$ was achieved by I. Maximoff (see [40, 41]), mentioned in the ending of Professor's letter:

   **Theorem B.** *Bounded function $f\colon (0,1) \to \mathbb{R}$ is equivalent in the Lebesgue sense to the derivative if and only if $f$ is of the first Baire class and, concurrently, satisfies the Darboux condition.*

   This theorem, together with the beautiful proof based on the David Preiss conception, can be found in fourth chapter of monograph [25]. Funnily enough, this proof effectively uses the Neugebauer theorem characterizing the (bounded) derivatives, mentioned in Professor's letter. Authors of monograph [25] call also in question the correctness of the original, i.e. given by I. Maximoff, proof of Theorem B (if it is true, then the author of the first correct proof of this theorem would be the already mentioned D. Preiss).
   Moreover, as it was shown in [25] after Goffman and Neugebauer, the characterization from Theorem B holds also in the class of bounded functions $f\colon (0,1) \to \mathbb{R}$, equivalent in the Lebesgue sense to the approximative derivative.
   More outlined information devoted to the characterizations of derivatives can be found in papers [15, 23] and [19]. Especially in the latter the following interesting result achieved by Krzysztof Chris Ciesielski is given [19, 53]:

   **Theorem C.** *Neither of the following function classes is topologicable: class $\Lambda$ of all derivatives of the Zahorski class $\mathcal{M}_k$, $k = 1, \ldots, 5$, class of all functions satisfying the Darboux condition, class of all measurable functions and class of all functions possessing the Baire property.*

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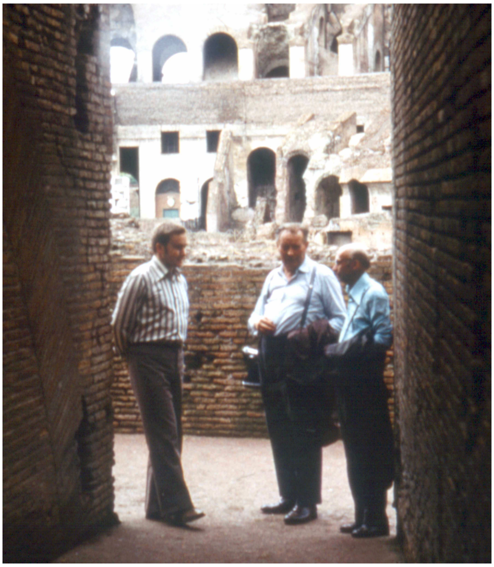

From left: Władysław Wilczyński, Zygmunt Zahorski and Jan Lipiński

Zygmunt Zahorski and Jan Lipiński